Föreläsning 14/11-13

G & S kap.6 Markov Chains

Section 6.1 – yesterday definition of Markov process $(X_n, n \geq 0)$

$$p_{ij} = P(X_{n+1}=j \mid X_n = i) = P(X_{n+1}=j \mid X_n = i, X_{n-1}=x_{n-1}, \ldots, X_0 = x_0)$$

$$p_{ij}^{(m)} = P(X_{n+m}=j \mid X_n=i) = P(X_{n+m}=j \mid \text{—} \shortparallel \text{———}), m \geq 1$$

$$P = (p_{ij})_{ij} \qquad P^{(m)} = (p_{ij}^{(m)})_{ij} = P^m$$

$\mu^{(n)}$ row matrix, $(\mu^{(n)})_i = P(X_n = i)$

$$\mu^{(n)} = \mu^{(0)} P^{(n)} = \mu^{(0)} P^n, \quad \mu^{(m+n)} = \mu^{(n)} P^m$$

example

Simple random walk

$X_n = \sum_{i=1}^{n} Y_i$ where $Y_1, Y_2, \ldots$ are IID r.v.'s with $\begin{cases} P(Y_i=-1)=q=1-p \\ P(Y_i=1)=p \end{cases}$

Markov chain.

6.2 Classification of states (values)

j is transient if $P(X_n = j \text{ for some } j \geq 1 \mid X_0 = j) < 1$

j is recurrent/persistent if $P(X_n = j \text{ for some } j \geq 1 \mid X_0 = j) = 1$

$T_j = \min(n \geq 1 : X_n = j)$

$\mu_j = E(T_j \mid X_0 = j) = \begin{cases} \infty & \text{we call } j \text{ null-persistent} \\ < \infty & \text{we call } j \text{ non-null-persistent} \end{cases}$

for j persistent
(if not it would result in ∞)

Some more notations

$$P_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}^{(n)} z^n, \quad p_{ij}^{(0)} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$$

$$F_{ij}(z) = \sum_{n=0}^{\infty} f_{ij}^{(n)} z^n \quad \text{where} \begin{cases} f_{ij}^{(n)} = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i), n \geq 1 \\ f_{ij}^{(0)} = 0 \end{cases}$$

Thm 6.2.3 $\quad P_{ii}(z) = 1 + F_{ii}(z) P_{ii}(z)$

$$P_{ij}(z) = F_{ij}(z) P_{jj}(z), \quad i \neq j$$

Corollary 6.2.4 *j is persistent iff $\sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty$ and then

forts. →

$$\sum_{n=0}^{\infty} P_{ij}^{(n)} = \infty \text{ whenever } f_{ij} > 0$$

$$\left(\text{where } f_{ij} = \sum_{n=0}^{\infty} f_{ij}^{(n)} = \sum_{n=1}^{\infty} f_{ij} = P(X_n = j \text{ some } n \geq 1 \mid X_0 = i) \right)$$

*j is transient iff $\sum_{n=0}^{\infty} P_{jj}^{(n)} < \infty$ and then $\sum_{n=0}^{\infty} P_{ij}^{(n)} < \infty \;\forall i$

Corollary 6.2.5 j transient $\Longrightarrow P_{ij}^{(n)} \to 0$ as $n \to \infty$

example forts.

Task: Check if a state j is recurrent or transient using Cor. 6.2.5 Enough checking if 0 is recurrent or transient.

$$P_{00}^{(n)} = \begin{cases} \binom{2k}{k} p^k q^k & \text{for } n = 2k \\ 0 & \text{for } n = 2k+1 \end{cases}$$

Binomial: $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

Consequently (by Cor 6.2.4) $\begin{cases} j \text{ transient if } \sum_{k=0}^{\infty} \binom{2k}{k} p^k q^k < \infty \\ j \text{ persistent if } \text{— '' —} = \infty \end{cases}$

Sterling formula: $k! \approx \frac{1}{\sqrt{2\pi}} k^{k+1/2} e^{-k}$ for big k.

$$\binom{2k}{k} p^k q^k = \frac{(2k)!\, p^k q^k}{(k!)(k!)} \approx \{\text{sterling}\} \approx \frac{\sqrt{2\pi}\,(2k)^{2k+1/2}\, p^k q^k \cancel{e^{-2k}}}{k^{k+1/2}\, k^{k+1/2}\, \cancel{e^{-k}}\, \cancel{e^{-k}}} =$$

$$= \frac{2\sqrt{\pi}\,(4pq)^k}{\sqrt{k}} \begin{cases} \text{some of these convergent for } p \neq q \text{ because } pq < 1/4 \\ \text{some of these divergent for } p = q = 1/2 \end{cases}$$

Proof 6.2.3 $P_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} P_{jj}^{(n-k)} \quad n \geq 1$

$$P_{ij}^{(n)} = (f_{ij} * P_{ij})(n)$$

$$P_{jj}(z) = \sum_{n=0}^{\infty} P_{jj}^{(n)} z^n = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} f_{jj}^{(k)} P_{jj}^{(n-k)} z^n = 1 + \underbrace{\sum_{n=1}^{\infty} f_{jj}^{(k)} z^k \underbrace{\left(\sum_{n=k}^{\infty} P_{jj}^{(n-k)} z^{n-k} \right)}_{P_{jj}(z)}}_{F_{jj}(z)}$$

$$= P_{jj}(z) F_{jj}(z) + 1$$

$i \neq j$: $P_{ij}(z) = \sum_{n=0}^{\infty} P_{ij}^{(n)} z^n = 0 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{ij}^{(k)} P_{jj}^{(n-k)} z^n = \ldots =$

$$= P_{jj}(z) F_{ij}(z)$$

∎

Proof 6.2.4 $f_{jj} = P(X_n = j \text{ some } n \geq 1 \mid X_0 = j) = \sum_{n=0}^{\infty} f_{jj}^{(n)} = F_{jj}(z=1) =$

$$= \frac{P_{jj}(1)-1}{P_{jj}(1)} = \frac{\sum_{n=0}^{\infty} P_{jj}^{(n)} - 1}{\sum_{n=0}^{\infty} P_{jj}^{(n)}} = \begin{cases} 1 & \text{for } \sum_{n=0}^{\infty} P_{jj}^{(n)} = \infty \\ <1 & \text{for } \sum_{n=0}^{\infty} P_{jj}^{(n)} < \infty \end{cases} \quad \blacksquare$$

Proof 6.2.5 By inspection. $\blacksquare$

Thm A persistent state i is null iff $P_{ii}^{(n)} \to 0$ as $n \to \infty$
If this holds, then $P_{ji}^{(n)} \to 0$ as $n \to \infty$ $\forall j$.

Definition
The period $d(i)$ of a state i is defined $d(i) = \underbrace{GCD}_{\text{största gem. nämnare}}(n \geq 1 : P_{ii}^{(n)} > 0)$

6.3 Classification of chain

G & S, i communicates with j $(i \to j)$ if $P_{ij}^{(n)} > 0$ some $n \geq 1$
(Hsu, j accesible from i)

G & S, i and j intercommunicates $(i \leftrightarrow j)$ if $i \to j$ and $j \to i$.
(Hsu, i and j communicates)

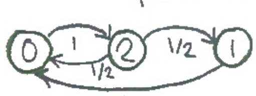
Chain is irreducible if all states intercommunicate $(i \leftrightarrow j)$

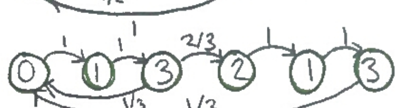
example
Possible values (= state space) = $\{0,1,2,3\}$

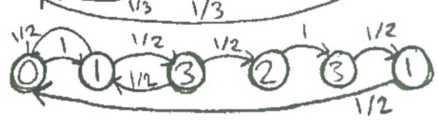
a) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix} = P$ b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \end{pmatrix} = P$ c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix} = P$

(period = minsta gemensamma delare som de # steg varje sätt att ta sig tillbaka på har.)

For each chain, which states intercommunicate and what are their periods? (What about transience and recurrence?)

a)  (intercommunicate) period = 1

b)  (intercommunicate) period = 3

c)  (intercommunicate) Period = 2